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DAVID W. TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

Bethesda, Maryland 20084

GENERATION OF ORTHOGONAL BOUNDARY-FITTED COORDINATE SYSTEMS

by

Henry J. Haussling
Roderick M. Coleman

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COMPUTATION, MATHEMATICS AND LOGISTICS DEPARTMENT
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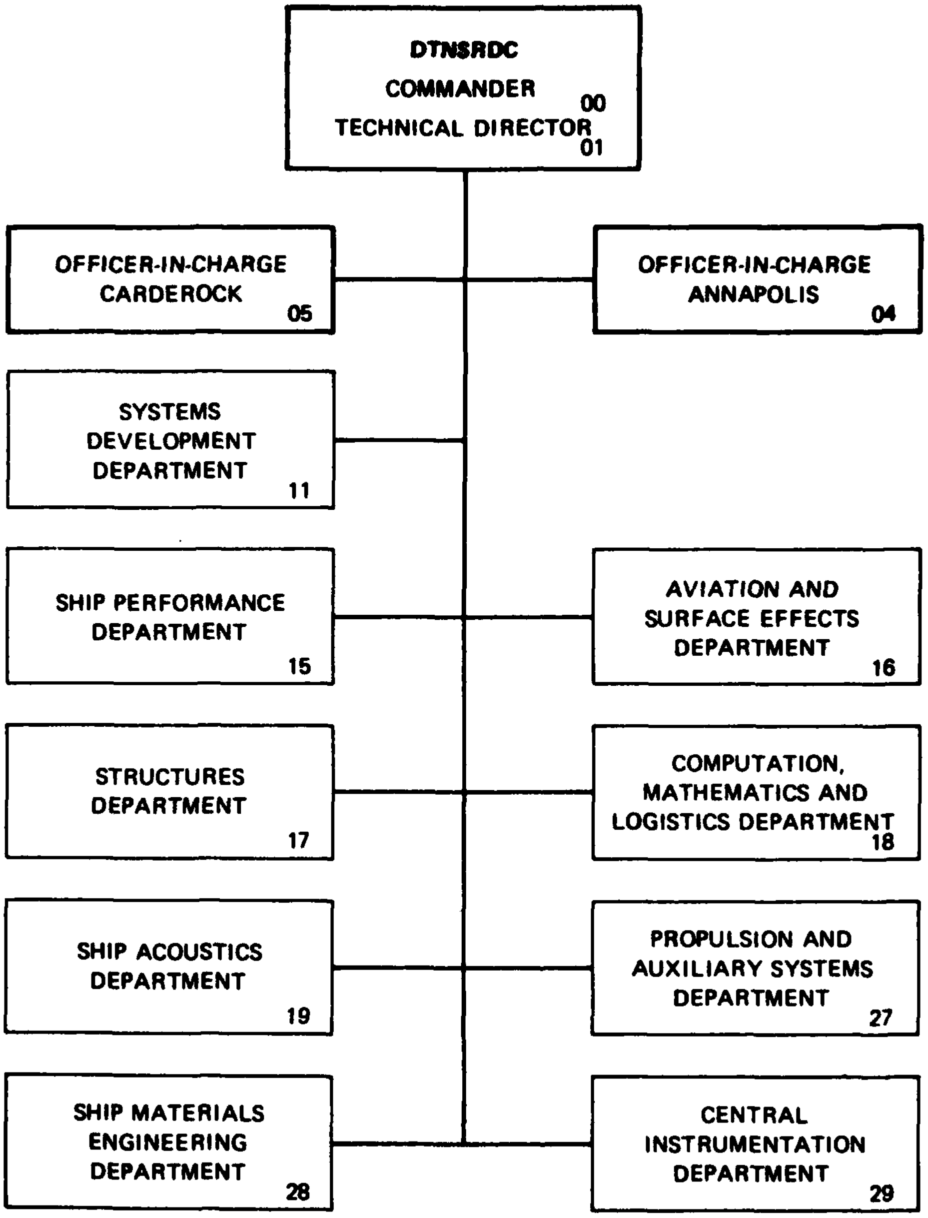

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Various methods for numerically generating orthogonal boundary-fitted coordinate systems are discussed. Simple generating equations are used with coordinate distributions specified on all of the boundaries. Iterative determination of orthogonalizing source terms for Poisson generating system yields better results and is applied successfully to several geometries.

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TABLE OF CONTENTS

LIST OF FIGURES

Page

ABSTRACT

Various methods for numerically generating orthogonal boundary-fitted coordinate systems are discussed. Simple generating equations are used with coordinate distributions specified on all of the boundaries. Iterative determination of orthogonalizing source terms for Poisson generating equations converges very slowly. An iterative solution of a new generating system yields better results and is applied successfully to several geometries.

ADMINISTRATIVE INFORMATION

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INTRODUCTION

Conformal mappings which are analytic functions of a complex variable have found extensive application to problems in physics. However, these mappings are restricted to two dimensions and have other limitations which sometimes seriously diminish their usefulness.

In recent years more general transformations have come into use under the names of "boundary-fitted" or "surface-oriented" coordinate systems.[1,2*] These coordinate systems are computed as approximate numerical solutions to elliptic generating equations. Usually the coordinate distribution on the boundaries of the physical regions can be specified arbitrarily. The standard conformal system is obtained only when the Laplace equation is used with a particular boundary coordinate distribution. Thus boundary-fitted coordinates encompass a much larger set of systems (including three-dimensional ones) than the subset of conformal mappings. However, these boundary-fitted systems are not, in general, orthogonal. This

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*A complete listing of references is given on page 29.

nonorthogonality can have a deleterious effect on accuracy, stability, and computational complexity.

Some effort has been devoted to finding an intermediate set of coordinate systems which retain most of the flexibility of the general boundary-fitted systems but yet are orthogonal. Potter and Tuttle[3] and Ghia, Hodge, and Hankey[4] presented a method for orthogonalizing a nonorthogonal boundary-fitted system. Other methods for generating orthogonal systems were presented in a note by Mobley and Stewart[5] and in work reviewed by Eisemann.[6] However, in all cases the specification of coordinate distribution was somehow restricted on at least one boundary. The present report discusses various methods for creating orthogonal coordinate systems using simple generating equations. These methods allow coordinate distributions to be specified arbitrarily on all of the boundaries. Iterative determination of orthogonalizing source terms for Poisson generating equations is attempted with very limited success. The iterative solution of a new generating system yields better results and successful application to several geometries is presented.

METHODS FOR CONSTRUCTING ORTHOGONAL SYSTEMS

Many investigators have used the system of Poisson equations

$$\begin{aligned} \xi_{xx} + \xi_{yy} &= P \\ \eta_{xx} + \eta_{yy} &= Q \end{aligned} \tag{1}$$

to generate numerical transformations from Cartesian coordinates (x,y) to the transformed coordinates (ξ,η) for a variety of geometries. The forcing functions P and Q in Equation (1) provide a means for influencing or controlling the coordinate system obtained.[1,2,4,7-12]

For ease in the numerical solution of Equation (1) all the calculations are done in the transformed plane on a uniform square mesh. Interchanging the dependent and independent variables gives

$$\begin{aligned} \alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} + J^2(Px_\xi + Qx_\eta) &= 0 \\ \alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} + J^2(Py_\xi + Qy_\eta) &= 0 \end{aligned} \tag{2}$$

where

$$\begin{aligned} \alpha &= x_\eta^2 + y_\eta^2 \qquad & \beta &= x_\xi x_\eta + y_\xi y_\eta \\ \gamma &= x_\xi^2 + y_\xi^2 \qquad & J &= x_\xi y_\eta - x_\eta y_\xi \end{aligned} \tag{3}$$

Replacing Equation (2) with central difference formulae yields

$$\begin{aligned} \begin{Bmatrix} x_{i\,j} \\ y_{i\,j} \end{Bmatrix} = [(\alpha_{i\,j} + J^2_{i\,j}\,P_{i\,j}/2) & \begin{Bmatrix} x_{i+1,j} \\ y_{i+1,j} \end{Bmatrix} \\ + (\alpha_{i\,j} - J^2_{i\,j}\,P_{i\,j}/2) & \begin{Bmatrix} x_{i-1,j} \\ y_{i-1,j} \end{Bmatrix} + (\gamma_{i\,j} + J^2_{i\,j}\,Q_{i\,j}/2) \begin{Bmatrix} x_{i,j+1} \\ y_{i,j+1} \end{Bmatrix} \\ + (\gamma_{i\,j} - J^2_{i\,j}\,Q_{i\,j}/2) & \begin{Bmatrix} x_{i,j-1} \\ y_{i,j-1} \end{Bmatrix} - (\beta_{i\,j}/2) \begin{Bmatrix} x_{i+1,j+1} + x_{i-1,j-1} \\ y_{i+1,j+1} + y_{i-1,j-1} \end{Bmatrix} \\ & \begin{Bmatrix} - x_{i-1,j+1} - x_{i+1,j-1} \\ - y_{i-1,j+1} - y_{i+1,j-1} \end{Bmatrix}]/2(\alpha_{i\,j} + \gamma_{i\,j}) \end{aligned} \tag{4}$$

where α_{ij}, β_{ij}, γ_{ij} and J_{ij} are central difference approximations to the corresponding variables of Equation (3).

Extensive use has been made of the Poisson and Laplace ($P \equiv Q \equiv 0$) generating systems of Equation (1) in the solution of fluid flow problems.[1,2,4,7] Several investigators[8-10] used simple exponential forcing functions to attract lines into or repel lines from certain regions of the physical plane. Thompson[11] and

Plant[12] have suggested a method for an a priori determination of P and Q which will retain boundary spacing throughout the region.

In this report we discuss automatic methods for finding functions P and Q which assure an orthogonal mesh for a given geometry. The condition for orthogonality, ξ = constant lines perpendicular to η = constant lines, is $\beta = 0$, since

$$\beta = 0 \Rightarrow \quad x_\xi / y_\xi = -y_\eta / x_\eta$$

which is equivalent to

$$1/y_x\big|_{\eta=\text{constant}} \qquad = -y_x\big|_{\xi=\text{constant}}$$

That is, the slopes of the two sets of coordinate lines are negative reciprocals of each other. An orthogonal coordinate system generated in this manner would be a solution of Equation (2) without the cross derivative terms, i.e.,

$$\begin{aligned} \alpha x_{\xi\xi} + \gamma x_{\eta\eta} + J^2(Px_\xi + Qx_\eta) &= 0 \\ \alpha y_{\xi\xi} + \gamma y_{\eta\eta} + J^2(Py_\xi + Qy_\eta) &= 0 \end{aligned} \tag{5}$$

Two important questions arise: Under what conditions do orthogonalizing source terms P and Q exist? If they exist, are they unique? Since we do not now have the answers to these questions, it is very important to have a test problem for which an orthogonal grid is known to exist. Failure to generate an orthogonal grid for such a problem can then safely be attributed to the method used.

A useful test problem is illustrated in Figures 1 and 2. These figures show a rectangular region with uniform boundary coordinate spacing in the horizontal direction and nonuniform boundary coordinate spacing in the vertical direction. The grid shown in Figure 1 was obtained by solving Equation (4) with $P \equiv Q \equiv 0$ using successive overrelaxation (SOR). Note that one characteristic of a Laplace-

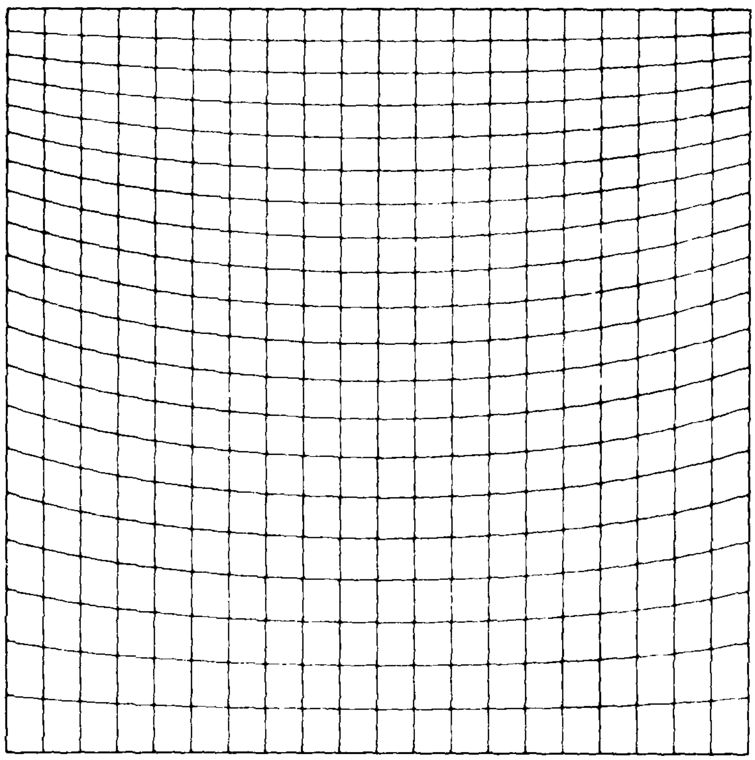

Figure 1 - Laplace Coordinate System with Unequal Spacing on Two Boundaries

generated mesh is that the coordinate lines tend toward equal spacing in the interior for any boundary spacing. This effect can be seen in Figure 1 near the center of the region.

An orthogonal grid for this boundary spacing can be generated using Equation (2) with source terms determined by the boundary coordinate distribution as follows:

$$P_{ij} = (-x_{\xi\xi}/x_\xi^3)_{i,1} \quad \text{and} \quad Q_{ij} = (-y_{\eta\eta}/y_\eta^3)_{1,j} \tag{6}$$

where i = 1 and j = 1 correspond to vertical and horizontal boundaries. This form for the forcing functions is derived by Plant[12] following a method suggested by Thompson[11] and Ghia.[4] The orthogonal grid thus generated is shown after 40 SOR iterations in Figure 2. (In this example $P \equiv 0$ and $Q = \text{const.} \neq 0$.) We have tested methods for generating orthogonal grids for their ability to reproduce the coordinate system of Figure 2.

One method of generating a transformation which yields an orthogonal coordinate system is to solve the systems of Equations (2) and (5) simultaneously. These systems represent four equations in the four unkowns, x, y, P, and Q. A solution, if it exists, would satisfy $\beta = 0$.

Unfortunately, solving the combined system of Equations (2) and (5) is not easy, but iterative methods are applicable. For instance, the finite-difference versions of Equation (2) can be used to obtain new estimates of x and y using current values of P and Q. Then Equation (5) in finite-difference form can be used to find new estimates of P and Q using the updated values of x and y. Solving Equation (5) for P and Q yields

$$\begin{aligned} P^n &= [\alpha(x_\eta y_{\xi\xi} - y_\eta x_{\xi\xi}) + \gamma(x_\eta y_{\eta\eta} - y_\eta x_{\eta\eta})]/J^3 \\ Q^n &= [\alpha(y_\xi x_{\xi\xi} - x_\xi y_{\xi\xi}) + \gamma(y_\xi x_{\eta\eta} - x_\xi y_{\eta\eta})]/J^3 \end{aligned} \tag{7}$$

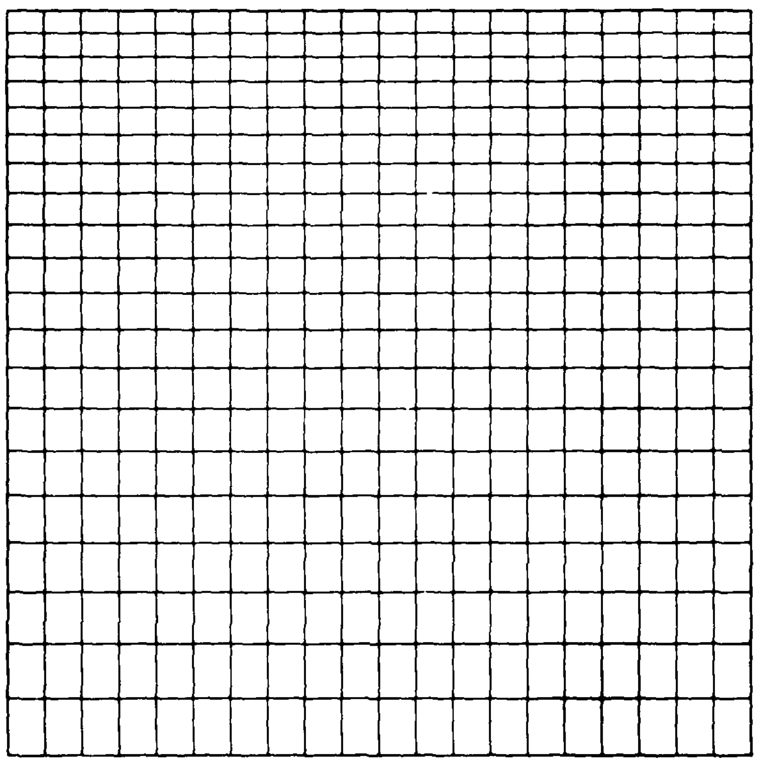

Figure 2 - Poisson Coordinate System with Unequal Spacing on Two Boundaries

where the superscripts indicate iteration level. Solving Equation (2) for P^{n-1} and Q^{n-1} yields

$$\begin{aligned} P^{n-1} &= [\alpha(x_\eta y_{\xi\xi} - y_\eta x_{\xi\xi}) - 2\beta(x_\eta y_{\xi\eta} - y_\eta x_{\xi\eta}) + \gamma(x_\eta y_{\eta\eta} - y_\eta x_{\eta\eta})]/J^3 \\ Q^{n-1} &= [\alpha(y_\xi x_{\xi\xi} - x_\xi y_{\xi\xi}) - 2\beta(y_\xi x_{\xi\eta} - x_\xi y_{\xi\eta}) + \gamma(y_\xi x_{\eta\eta} - x_\xi y_{\eta\eta})]/J^3 \end{aligned} \tag{8}$$

Comparison of Equations (7) and (8) shows that

$$P^n = P^{n-1} + 2\beta(x_\eta y_{\xi\eta} - y_\eta x_{\xi\eta})/J^3 \tag{9a}$$

$$Q^n = Q^{n-1} + 2\beta(y_\xi x_{\xi\eta} - x_\xi y_{\xi\eta})/J^3 \tag{9b}$$

Thus, solving Equations (2) and (5) iteratively is equivalent to using Equations (2) and (9) in an iterative scheme. Note that, according to Equation (9), if $\beta = 0$, then $P^n = P^{n-1}$ and $Q^n = Q^{n-1}$ and the iteration process for P and Q will have converged.

Unfortunately, the system formed by Equations (2) and (9) cannot handle the test problem. This can be demonstrated by considering what happens in an attempt to achieve the orthogonal coordinate system of Figure 2 by iterating according to Equations (2) and (9), with the coordinate system of Figure 1 as a starting point. Consider the local grid configurations in Figures 3(a) and 3(b), both of which appear in Figure 1. Equation (9) predicts an initial Q^1 ($Q^o = 0$) of different sign for the two configurations, even though the ultimate Q desired is, according to Equation (6), uniform over the region. Thus iterating according to Equation (9) leads to deterioration of the system in part of the region, and the calculations diverge. This failure is not of the basic system of Equations (2) and (5) but rather of the iterative method of solving the system.

Further study of Figure 3 can lead to an improved method. Equation (9) represents an adjustment of the source term Q_{ij} depending on β_{ij}. Yet the

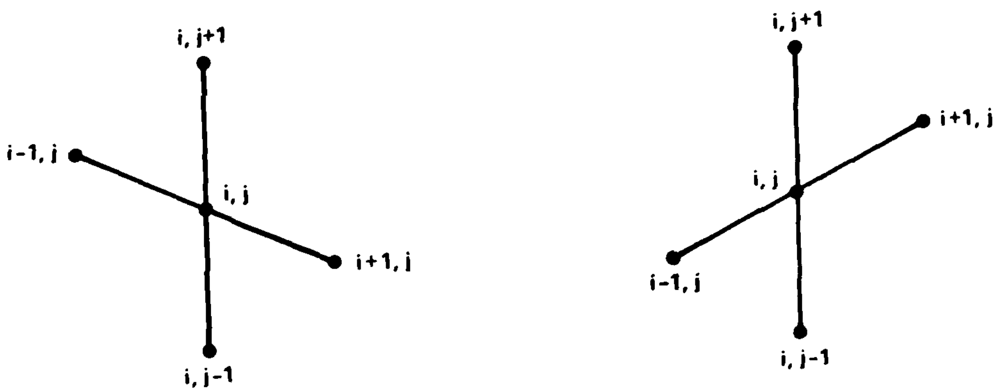

Figure 3a - $\beta < 0$ Figure 3b - $\beta > 0$

Figure 3 - Local Grid Configuration

configurations in Figure 3 do not, by themselves, yield any information about the desired movements of the grid point (i, j). Rather, Figure 3(a) suggests a movement of the point (i-1,j) downward and (i+1,j) upward to improve orthogonality. This leads to a modification of Equation (9b) of the form

$$Q^{n+1}_{i-1,j} = Q^{n}_{i-1,j} + \beta_{ij}\left|(y_\xi x_{\xi\eta} - x_\xi y_{\xi\eta})/J^3\right|_{i-1,j} \tag{10a}$$

$$Q^{n+1}_{i+1,j} = Q^{n}_{i+1,j} - \beta_{ij}\left|(y_\xi x_{\xi\eta} - x_\xi y_{\xi\eta})/J^3\right|_{i+1,j} \tag{10b}$$

which will move points in a desirable manner in both Figures 3(a) and 3(b). Replacing i by i+1 in Equation (10a) and by i-1 in Equation (10b) and then adding the results yields

$$Q^{n+1}_{ij} = Q^{n}_{ij} + \frac{\beta_{i+1,j} - \beta_{i-1,j}}{2}\left|(y_\xi x_{\xi\eta} - x_\xi y_{\xi\eta})/J^3\right|_{ij}$$

or

$$Q^{n+1}_{ij} = Q^{n}_{ij} + \beta_\xi\left|(y_\xi x_{\xi\eta} - x_\xi y_{\xi\eta})/J^3\right|_{ij} \tag{11a}$$

Equation (11a) seems to provide a workable scheme along with

$$P^{n+1}_{ij} = P^{n}_{ij} + \beta_\eta\left|(x_\eta y_{\xi\eta} - y_\eta x_{\xi\eta})/J^3\right|_{ij} \tag{11b}$$

which is arrived at through similar arguments.

The coordinate system shown in Figure 4, which is almost identical to that of Figure 2, was generated from the initial system of Figure 1 in 1500 iterations using Equations (2) and (11). Overrelaxation was used in the source term iteration in an unsuccessful attempt to accelerate the convergence.

A nonorthogonal mesh with nonuniform boundary coordinate spacing in both vertical and horizontal directions, as shown in Figure 5, was used as the initial guess in a further test of the system of Equations (2) and (11). In this case both P and

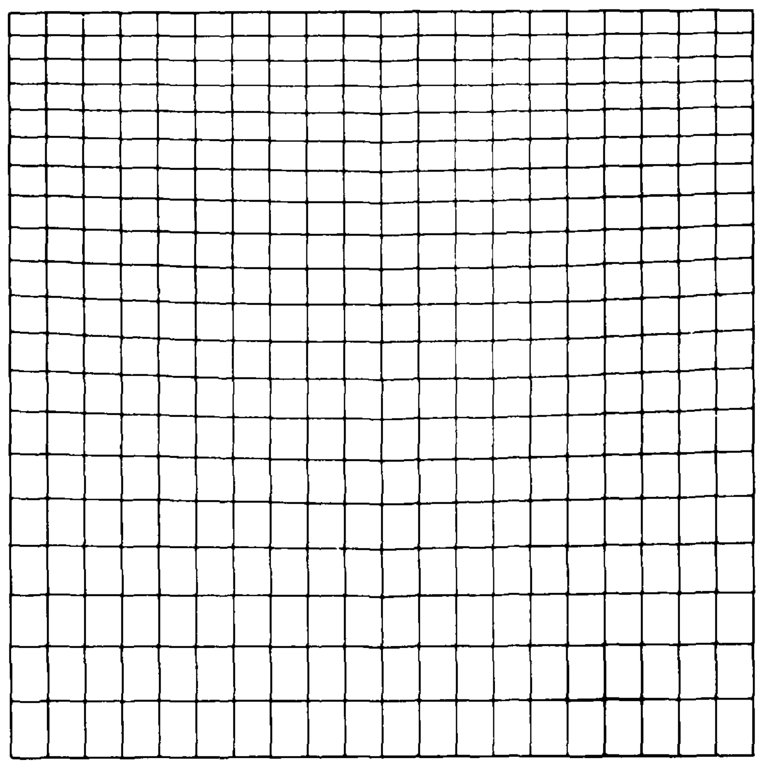

Figure 4 - Orthogonal Coordinate System with Unequal Spacing on Two Boundaries

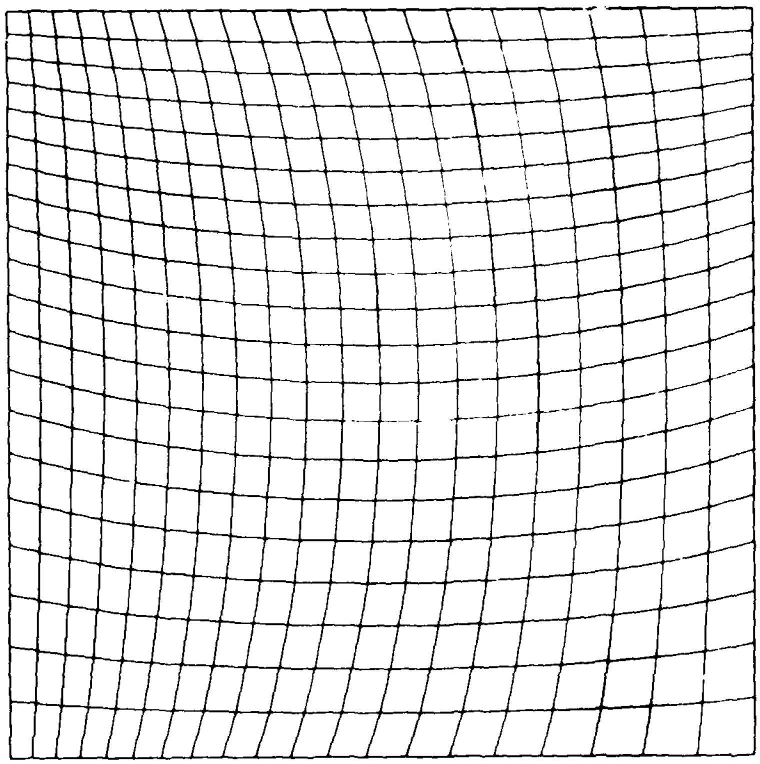

Figure 5 - Nonorthogonal Coordinate System with Unequal Spacing on all Boundaries

Q source terms had to be determined. The result after 1000 iterations is shown in Figure 6. The "waviness" of the mesh, even after so many iterations, indicates the convergence difficulties of this method which may preclude it in many cases as a useful scheme for generating orthogonal grids.

In applying Equations (2) and (11) we are attempting to find a coordinate system with source terms P and Q such that

$$\beta_{\xi} = \beta_{\eta} = 0 \tag{12}$$

The solution β = constant to Equation (12) exists only if it is consistent with the boundary data. Only at the corners of the computational region is β specified in advance. Thus we can hope for a solution if β is the same at all corners and for an orthogonal solution only when $\beta = 0$ at the corners.

A different approach to this orthogonal grid problem is to bypass the Poisson equations altogether and to choose a generating system based entirely on β. Once the grid points have been suitably positioned in the physical region, the forcing functions for the Poisson system that would generate this configuration could be found from Equation (7).

Even though $\beta = 0$ *is the condition for orthogonality, this equation alone is* not sufficient for obtaining the required transformation. Two equations are needed, since we must find both the x and y coordinates of the transformed points.

As a new generating system, consider Equation (12). Expanding Equation (12) gives

$$x_{\xi}x_{\xi\eta} + x_{\xi\xi}x_{\eta} + y_{\xi}y_{\xi\eta} + y_{\xi\xi}y_{\eta} = 0 \tag{13a}$$

$$x_{\xi}x_{\eta\eta} + x_{\xi\eta}x_{\eta} + y_{\xi}y_{\eta\eta} + y_{\xi\eta}y_{\eta} = 0 \tag{13b}$$

To compute the transformation, Equations (13a) and (13b) are combined as follows:

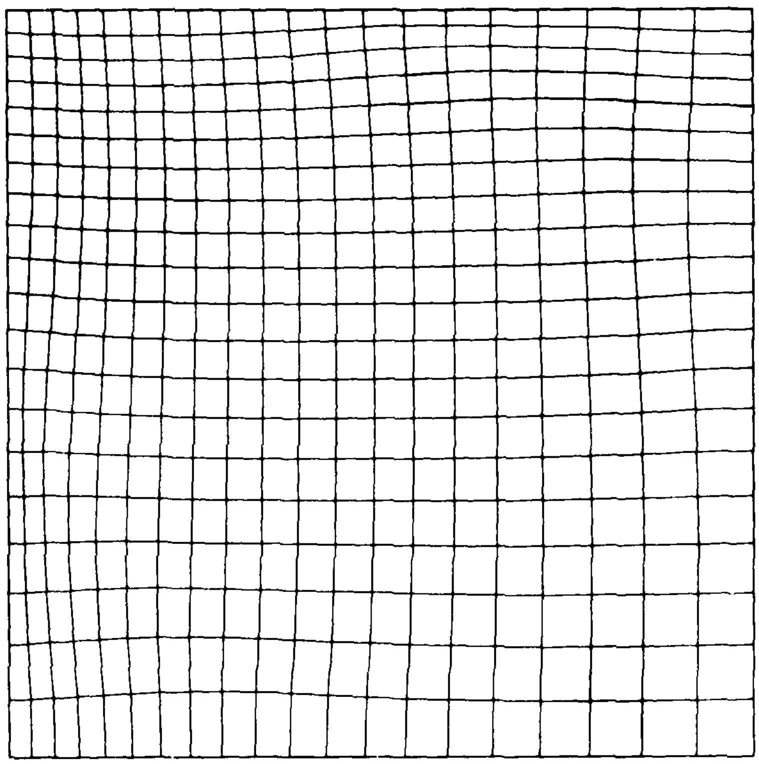

Figure 6 - Nearly Orthogonal Coordinate System with Unequal Spacing on all Boundaries

the product of Equation (13a) and x_η is added to the product of Equation (13b) and x_ξ, yielding

$$x_\eta^2\, x_{\xi\xi} + x_\xi^2\, x_{\eta\eta} + 2x_\xi x_\eta x_{\xi\eta} + x_\eta y_\eta y_{\xi\xi} + x_\xi y_\xi y_{\eta\eta} + (x_\eta y_\xi + x_\xi y_\eta) y_{\xi\eta} = 0 \tag{14a}$$

the product of Equation (13a) and y_η is added to the product of Equation (13b) and y_ξ, yielding

$$y_\eta^2\, y_{\xi\xi} + y_\xi^2\, y_{\eta\eta} + 2y_\xi y_\eta y_{\xi\eta} + x_\eta y_\eta x_{\xi\xi} + x_\xi y_\xi x_{\eta\eta} + (x_\eta y_\xi + x_\xi y_\eta) x_{\xi\eta} = 0 \tag{14b}$$

The reason for replacing Equations (13) with Equations (14) is to obtain a non-zero coefficient for x_{ij} and y_{ij} in the finite-difference forms of Equations (14a) and (14b). This eliminates the possibility of dividing by zero in the iteration process. Each derivative in Equations (14a) and (14b) is replaced by the appropriate central difference formula and the system is solved iteratively using successive overrelaxation.

The system of Equation (12) not only successfully generates the orthogonal grid of Figure 2 but also generates the acceptable coordinate system of Figure 7 from the initial system of Figure 5 in 500 iterations, half the number of iterations needed to generate the unacceptable system of Figure 6 with Equations (2) and (11).

FURTHER EXAMPLES

In each of the final six examples, we show a nonorthogonal coordinate system generated by Equation (2) with $P \equiv Q \equiv 0$ (a Laplace system) and a coordinate system generated by Equation (12). The first of these examples, shown in Figure 8, is a simply-connected region with one convex boundary. Next we have a similar region with a concave rather than convex curved boundary as seen in Figure 9. Note that

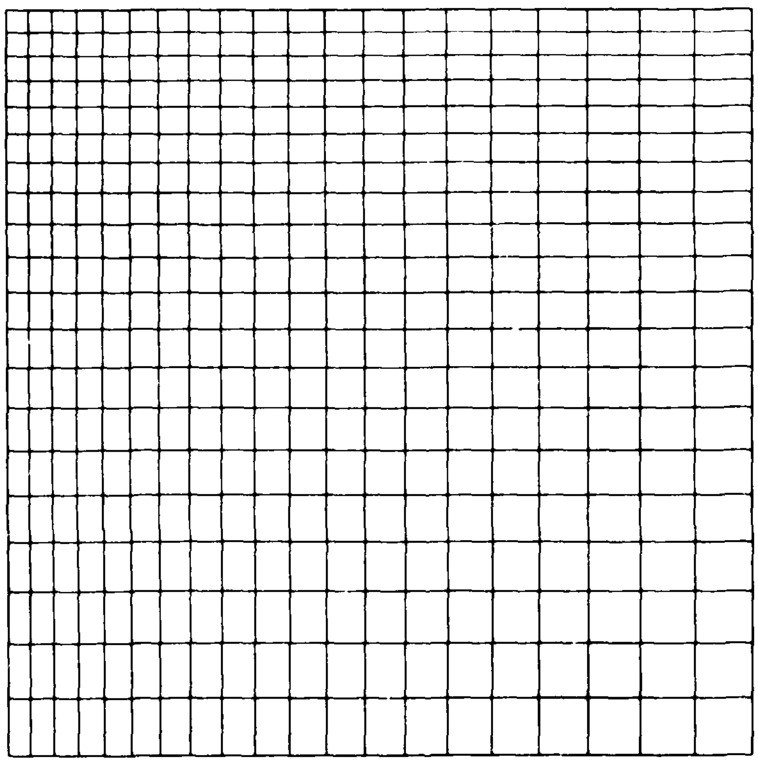

Figure 7 - Orthogonal Coordinate System Generated by $\beta_\xi = \beta_\eta = 0$

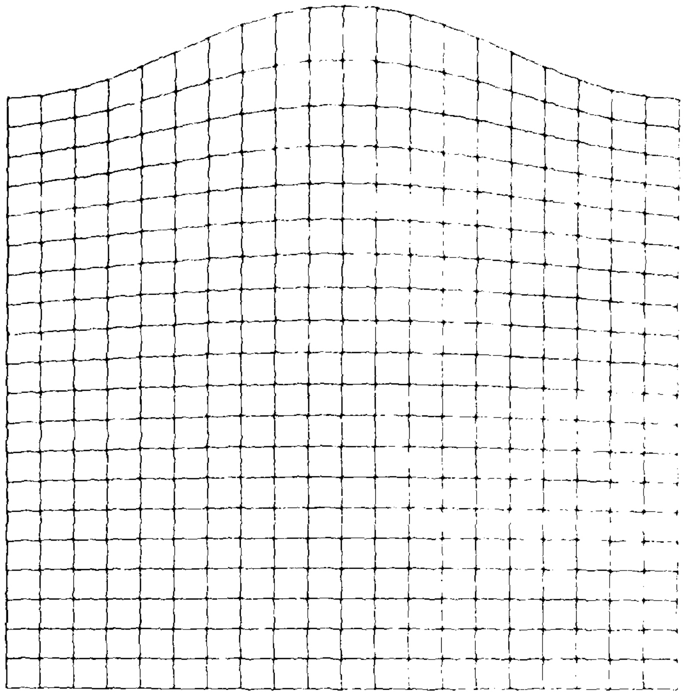

Figure 8a - Nonorthogonal

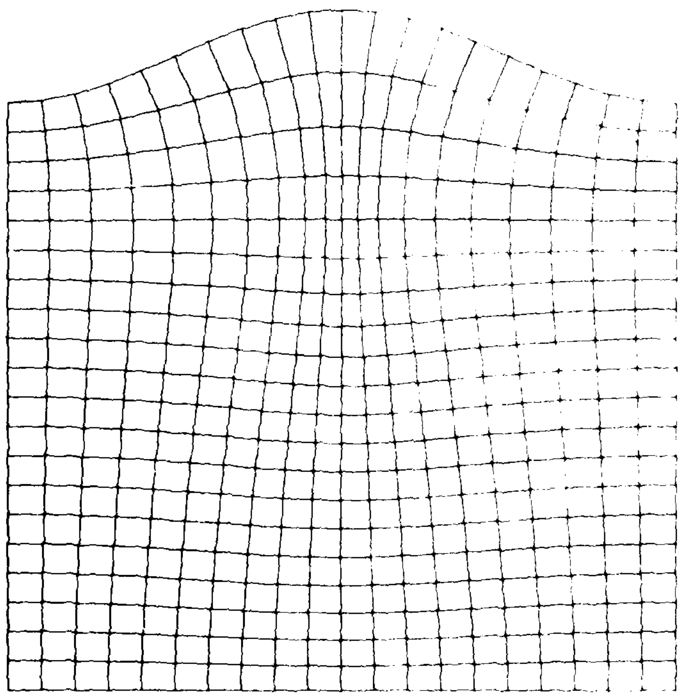

Figure 8b - Orthogonal

Figure 8 - Coordinate System for Region with Convex Boundary

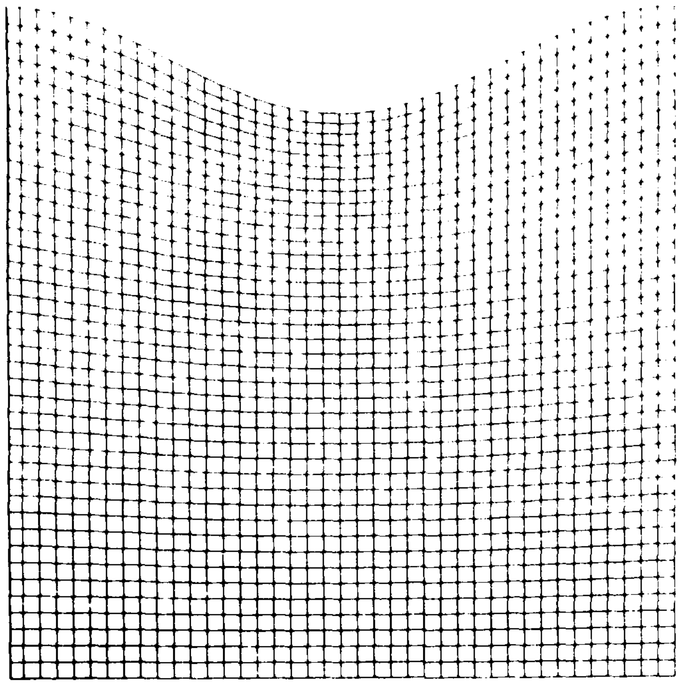

Figure 9a - Nonorthogonal

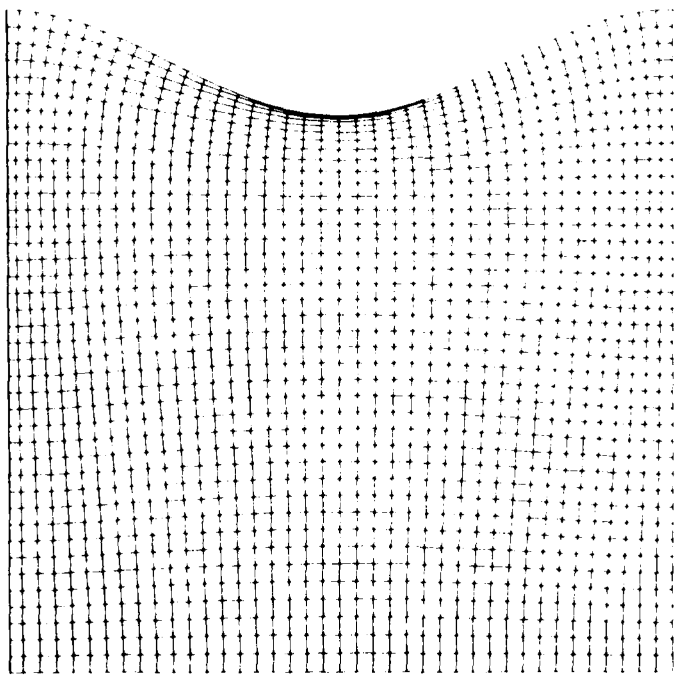

Figure 9b - Orthogonal

Figure 9 - Coordinate System for Region with Concave Boundary

the orthogonal mesh must have rather fine spacing near the concave upper boundary to accommodate the curvature. The fine mesh spacing raises a question concerning the possibility of a singularity in this vicinity. This is a valid question, one that needs to be considered each time the boundary-fitted coordinate technique is used since almost any generating system can produce unacceptable meshes for particular regions. To verify that the fine mesh spacing in Figure 9b does not indicate a singularity in the transformation, we have refined the mesh. Figure 10 compares two different grids, one coarse with 1681 points and the other fine with 6561 points, generated for the concave region. The fact that corresponding grid lines are in about the same position in both meshes confirms that the coarse discretization yields a good approximate solution to the exact problem. A further confirmation comes from consideration of the Jacobian at the midpoint of the upper boundary. The value of $J=0.00186$ computed on the coarse mesh is seen to agree very well with $J=0.00047$ on the fine mesh when it is taken into account that these quantities should differ by a factor of four because of the discretization details. There is no indication of a zero Jacobian in the region.

In order to demonstrate some of the problems that can arise, we attempted to generate an orthogonal mesh on a region similar to the previous one but with greater curvature of the concave boundary. The grid shown in Figure 11a was generated by Equation (2) with $P \equiv Q \equiv 0$ while the unacceptable grid in Figure 11b was generated by the system of Equation (12). Of course an unacceptable mesh such as this one with crossing lines indicates a singular transformation which can often lead to numerical difficulties. But problems like this can also arise from iterative schemes based on the Poisson system if the forcing functions are not chosen carefully. To verify this we computed directly forcing functions P and Q using Equation (2) and x and y given as in Figure 11b. We then solved Equation (2)

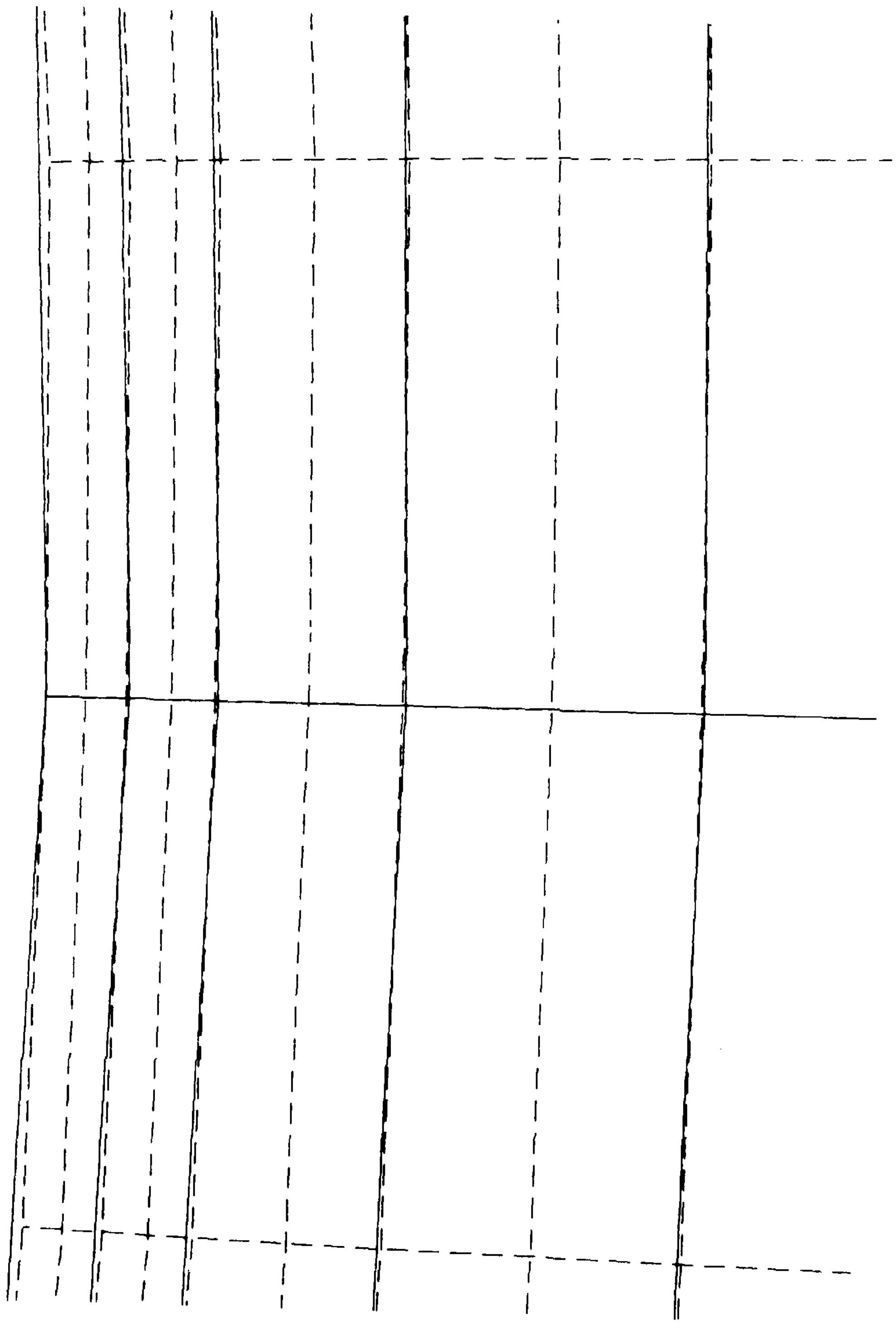

Figure 10 - Comparison of Orthogonal Grids for Region with Concave Boundary
—— 1681 Points, - - - 6561 Points

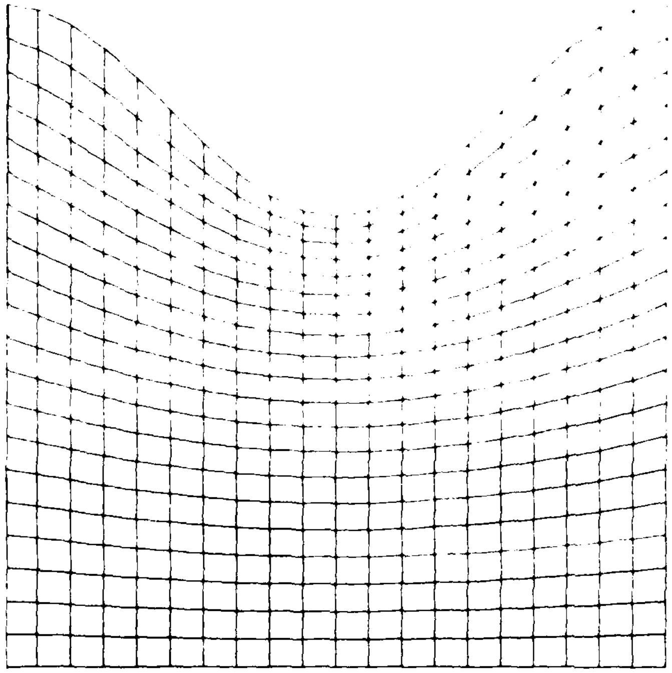

Figure 11a - Laplace Coordinate System

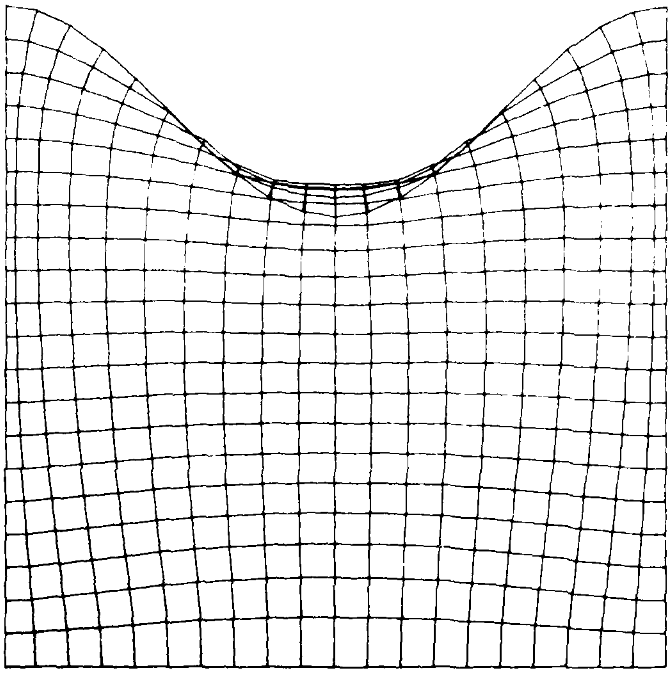

Figure 11b - Grid Lines Generated by $\beta_\xi = \beta_\eta = 0$

Figure 11 - Concave Region

iteratively for x and y using this P and Q, thus regenerating the grid of Figure 11b.

As the next example, consider a doubly-connected region bounded by concentric circles as shown in Figures 12 and 13. Since this region is symmetric to a line through the center, each grid was generated for half the region and reflected in the line of symmetry. The symmetry line was treated as a boundary with fixed coordinate distribution, thus assuring that $\beta = 0$ at the corners of the computational region. The spacing on the outer boundary, but not on the inner boundary, was uniform. Had the spacing on both boundaries been uniform, the grid produced by the Laplace generating system (Figure 12a) would have been the usual polar coordinate system which is orthogonal. In Figure 12b, the line of symmetry was taken as a horizontal line through the center of the figure while the line of symmetry for Figure 13 was a vertical line through the center. Interestingly, the two orthogonal grids thus generated (Figures 12b and 13) are quite dissimilar as a result of different points being held constant after the same initial guess.

For a final test of the method, we consider a square region with the coordinates distributed uniformly on three boundaries, nonuniformly on the fourth. Figure 14a shows a nonorthogonal grid with 441 points and Figure 14b a converged solution to Equation (12) with $\beta \approx 0.00020$ throughout the field except on the boundaries. In an attempt to improve the orthogonality, the number of grid points was increased to 1681 as seen in Figure 15. This refinement of the grid did not improve the orthogonality. Thus we believe that no orthogonal mesh exists for this configuration and that the existence of a numerical solution with a small nonzero β results from the weak connection between the interior grid points and the corner points, the only points where $\beta = 0$ is enforced.

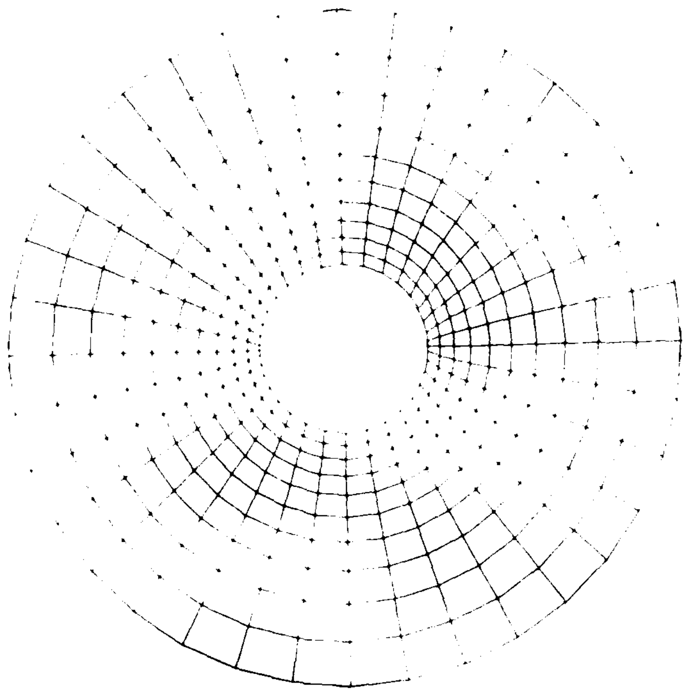

Figure 12a - Nonorthogonal

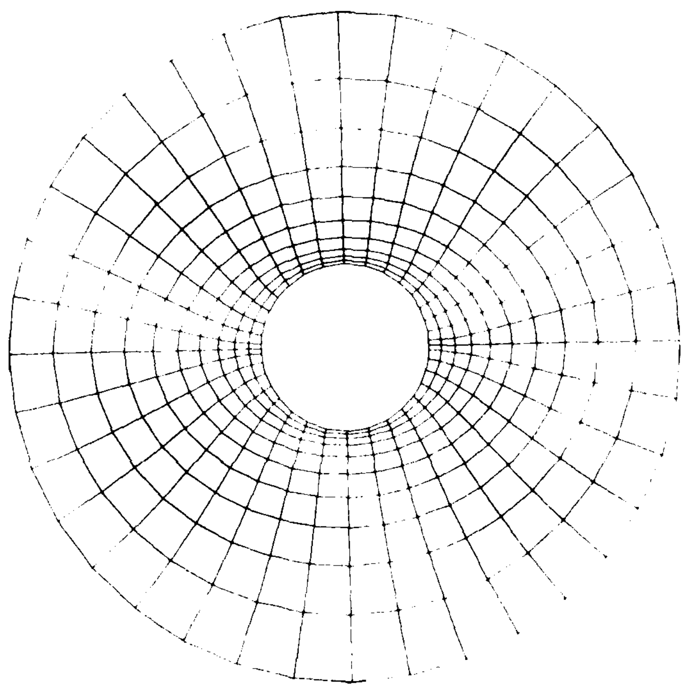

Figure 12b - Orthogonal

Figure 12 - Coordinate System for Annular Region

(Horizontal Symmetry Line)

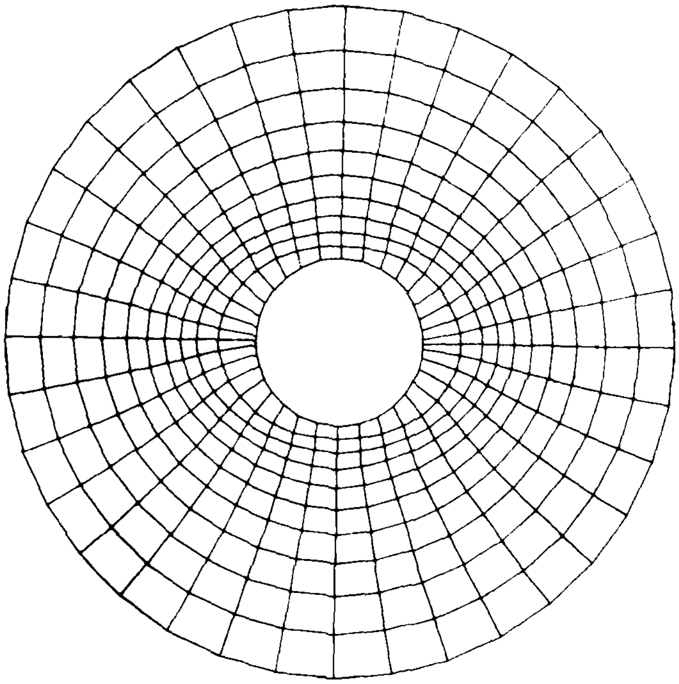

Figure 13 - Orthogonal Coordinate System for Annular Region
(Vertical Symmetry Line)

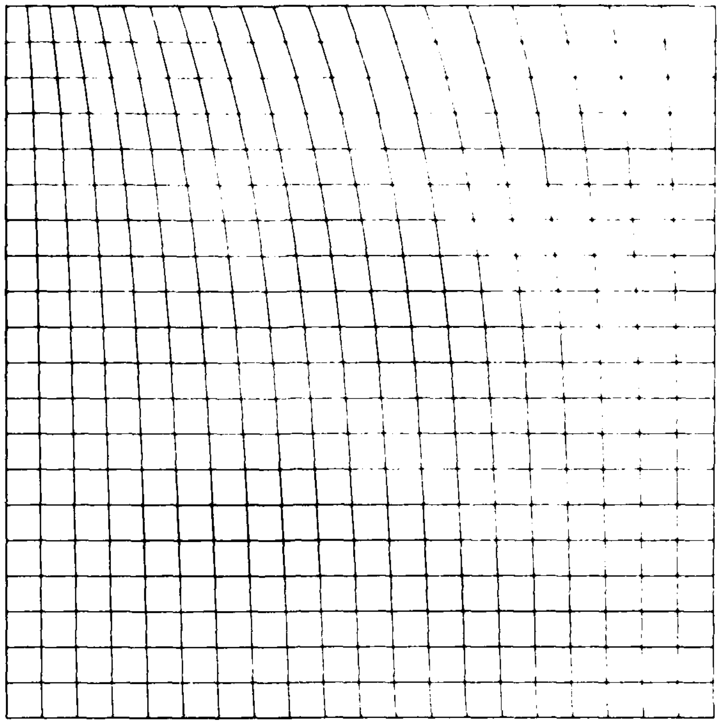

Figure 14a - Nonorthogonal

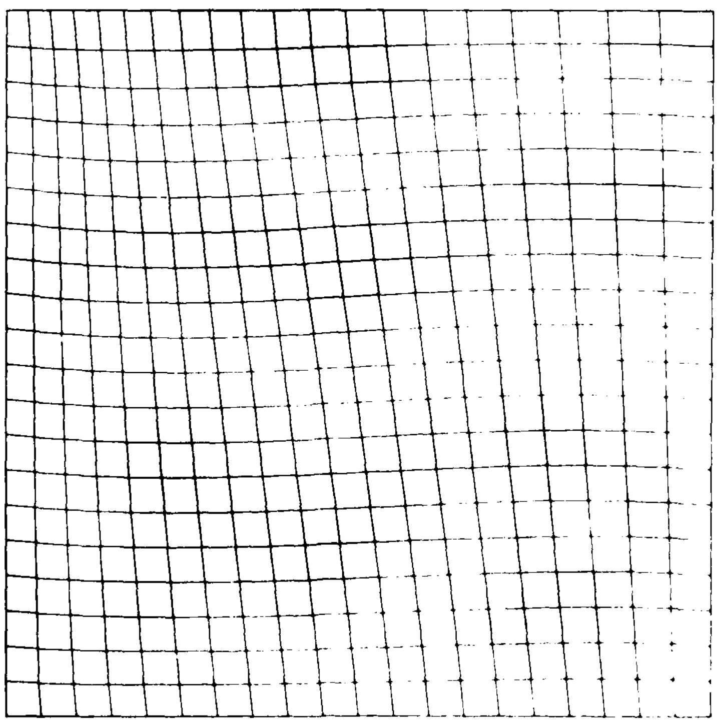

Figure 14b - Orthogonal

Figure 14 - Coordinate System with Unequal Spacing on One Boundary (441 Points)

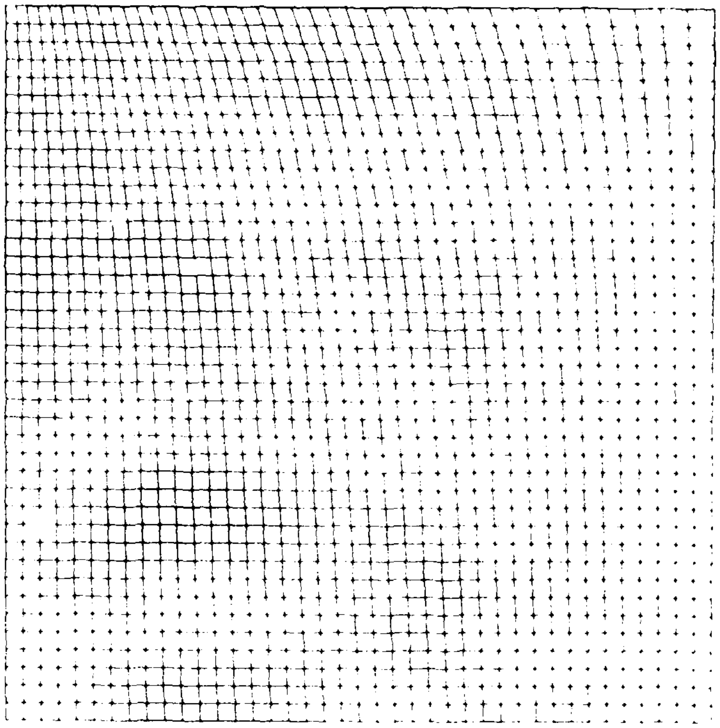

Figure 15a - Nonorthogonal

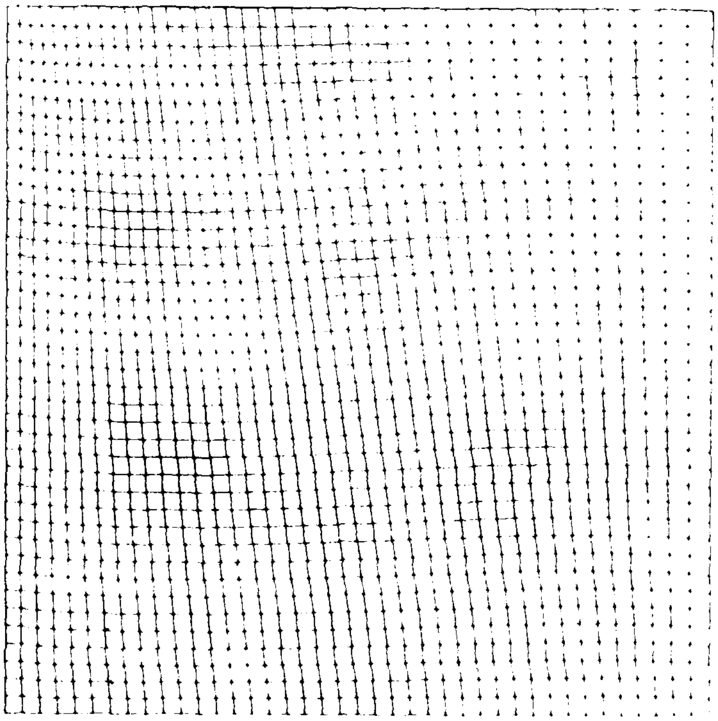

Figure 15b - Orthogonal

Figure 15 - Coordinate System with Unequal Spacing on One Boundary (1681 Points)

SUMMARY

Orthogonal boundary-fitted coordinate systems can be generated for a specified boundary coordinate distribution by the system of Equation (2) if P and Q can be found such that $\beta = 0$. One way to do this would be to solve the combined system of Equations (2) and (6). Iterative solution of this system using Equations (9) diverged on a simple test problem. An alternative iterative scheme given by Equations (2) and (11) converged extremely slowly for simple test problems. An iterative method based on the solution of Equation (12) rather than on the Poisson generating system had better convergence properties and was used to generate orthogonal boundary-fitted systems for several test problems.

While questions remain concerning the existence and uniqueness of orthogonal coordinate systems, and although there is much room for improvement in the methods for developing them, the generating methods presented here add to the available, useful techniques for constructing these systems. *Of course, it must be realized* that a particular orthogonal coordinate system is not necessarily better than any given nonorthogonal system. In general, it seems beneficial to require orthogonality whenever it does not lead to a serious loss of other desirable coordinate system properties.

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REFERENCES

1. Thompson, J.F. et al., "Use of Numerically Generated Body Fitted Coordinate Systems for Solution of the Navier-Stokes Equations," AIAA Second Computational Fluid Dynamics Conference, Hartford, Connecticut (June 19-20, 1975).

2. Ghia, U. et al., "Use of Surface-Oriented Coordinates in the Numerical Simulation of Flow in a Turbine Cascade," Lecture Notes in Physics, Vol. 59, p. 197, Springer-Verlag, Berlin/ Heidelberg/New York (1976).

3. Potter, D.E. and G.H. Tuttle, "The Construction of Discrete Orthogonal Coordinates," J. Comp. Phys. 13, 483 (1973).

4. Ghia, U. et al., "An Optimization Study for Generating Surface-Oriented Coordinates for Arbitrary Bodies in High-Re Flow," AFFDL-TR-77-117, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base (1977).

5. Mobley, C.D. and R.J. Stewart, "On the Numerical Generation of Boundary-Fitted Orthogonal Curvilinear Coordinate Systems," J. Comp. Phys. 34, 124-135 (1980).

6. Eiseman, P.R., "Geometric Methods in Computational Fluid Dynamics," ICASE Report No. 80-11 (1980).

7. Haussling, H.J. and R.M. Coleman, "Finite-Difference Computations Using Boundary-Fitted Coordinates for Free-Surface Potential Flows Generated by Submerged Bodies," Proc. 2nd Intl. Conf. Numerical Ship Hydrodynamics, Univ. of California, Berkeley (1977).

8. Thames, F.C. et al., "Numerical Solutions for Viscous and Potential Flow about Arbitrary Two-Dimensional Bodies Using Body-Fitted Coordinate Systems," J. Comp. Phys. 24, 245 (1977).

9. Haussling, H.J., "Boundary-Fitted Coordinates for Accurate Numerical Solution of Multibody Flow Problems," J. Comp. Phys. 30, 107 (1979).

10. Haussling, H.J. and R.M. Coleman, "Nonlinear Water Waves Generated by an Accelerated Circular Cylinder," J. Fluid Mech. 92, part 4, 767 (1979).

11. Thompson, J.F. and S.P. Shanks, "Numerical Solution of the Navier-Stokes Equations for 2-D Surface Hydrofoils," MSSU-EIRS-ASE-77-4, Engr. and Industrial Research Station, Mississippi State Univ., 1977.

12. Plant, T.J., "An Exact Velocity Potential Solution of Steady, Compressible Flow Over Arbitrary Two-Dimensional and Axisymmetric Bodies in Simply Connected Fields," AFFDL-TR-77-116, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, 1977.

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